MODEL OF A GLOBAL ELECTRIC CIRCUIT WITH CONDITIONS AT MAGNETIC CONJUGATE POINTS OF THE UPPER BOUNDARY OF THE ATMOSPHERE IN THE NON-STATIONARY CASE

National Research Lobachevsky State University of Nizhni Novgorod
(NNSU), Nizhny Novgorod, Russia
e-mail: natasha.denisova@mail.ru

Received December 19, 2023
Revised June 26, 2024
Accepted September 26, 2024

A new analytical representation of the electric potential is obtained for the classical non-stationary model of the global electrical circuit of the atmosphere, occupying a spherical layer, the conductivity of which increases exponentially along the radius. The boundary conditions of the model take into account the relationship between the values of the electric potential and current at magnetically conjugate points of the upper boundary of the atmosphere. Using the obtained representation, the potential distribution for a current dipole in a spherical layer is analyzed. New asymptotic formulas for the electric potential of a current dipole at $t\to\infty$ at each point of the spherical layer are obtained. An analytical expression for the Green's function of the corresponding initial-boundary value problem is found.

DOI: 10.31857/S00167940250207e9

1. INTRODUCTION

In the theory of atmospheric electricity, several mathematical models of the global electric circuit (GEC) have been proposed based on the study of the distribution of electric field potential induced by third-party currents modeling the separation currents in a thunderstorm cloud. Most of the mathematical models are reduced to finding the electric field potential from a boundary value problem (in the stationary case) or an initial-boundary value problem (in the non-stationary case) for a differential equation in some domain. In these problems, 1) the domain in which the problem is solved, 2) the function modeling the atmospheric conductivity, 3) the right-hand side of the equation modeling lightning generators, and 4) the boundary conditions may vary. The main results of the HEC theory and the literature review can be found in [Mareev, 2010; Mareev et al., 2019; Morozov, 2011].

In this paper, we consider a nonstationary classical HEC model for the atmosphere occupying a globular layer, where relations relating the values of electric potential and current at magnetically conjugate points are used as boundary conditions at the upper boundary of the atmosphere. Such boundary conditions for the stationary problem were used in [Hays and Roble, 1979; Ogawa, 1985]. The issues of mathematical correctness of problem formulations in both the stationary and nonstationary cases with such boundary conditions are discussed in [Kalinin and Slyunyaev, 2017].

A similar nonstationary problem with a simpler boundary condition was considered in [Morozov, 2005]. The main reason for using a simple boundary condition was the statement that in the lower atmosphere the upper boundary condition does not influence the solution.

In [Denisova and Kalinin, 2018], an attempt was made to compare analytical solutions of two corresponding stationary problems with different conditions at the upper boundary of the atmosphere. The paper shows that if the upper boundary of the globular layer is at a height greater than90км, then there is indeed a part of the globular layer containing current generators in which the values of the solutions of the two different boundary value problems are close. However, in the part of the globular layer located above the generators, these solutions are different. If the upper boundary of the ball layer is at a height lower than70км, the solutions of the problems are different in the whole ball layer, and especially in magnetically conjugate points at all heights. Since the solutions of the problems with different boundary conditions in the region above the current generators are different at any thickness of the globular layer, the study of the electric field distribution in the problem with boundary conditions [Hays and Roble, 1979] at the upper boundary of the atmosphere taking into account magnetically conjugate points is of interest.

The aim of the present work is the analytical solution of the initial boundary value problem for the potential in the atmosphere, the electric conductivity of which is exponentially increasing, with boundary conditions [Hays and Roble, 1979] at the upper boundary of the balloon layer. Finding the Green's function of the corresponding initial boundary value problem. Investigation of the current dipole potential distribution and derivation of asymptotic formulas at t .$\to\infty$

2. PROBLEM STATEMENT. NUMERICAL STUDY

The electric potential$\Phi(r,\theta,\varphi,t)$ of the atmosphere occupying the globular layer$r_0 < r < r_m$, when the vertical current is switched on at the initial moment, satisfies Eq:

$$\frac{1}{4\pi}\frac{\partial\Delta\Phi}{\partial t} + div(\sigma grad\Phi) = div\boldsymbol{j}^{ext}, \qquad (1)$$

boundary

$$\Phi(r,\theta,\varphi,t)|_{r=r_m} = \Phi(r,\pi-\theta,\varphi,t)|_{r=r_m,} \qquad (2)$$

$$\left.\frac{\partial\Phi(r,\theta,\varphi,t}{\partial r}\right|_{r=r_m} = -\left.\frac{\partial\Phi(r,\pi-\theta,\varphi,t}{\partial r}\right|_{r=r_m}, (3)$$

$$\Phi|_{r=r_0} = 0 (4)$$

and initial $$\Phi|_{t=0} = 0 \quad (5)$$

terms.

The paper further assumes that the electrical conductivity of the atmosphereσ depends only on the radius and increases exponentially with radius

$$\sigma(r) = \sigma_0 exp\left(\frac{r-r_0}{H}\right),$$

σ_0 —electric conductivity near the spherical Earth surface; *r* - distance from the Earth center;r_0 - Earth radius (the following values were used in numerical calculations:$r_0 = 6370$км$, H = 6$км; высотамагнитосферы$h_m = r_m - r_0, h_m = 100$км$); \boldsymbol{j}^{ext}$ - density of third-party electric currents created by lightning generators;r, θиφ - spherical coordinates.

We will consider a single third-party current source with the number *s*. In the case of several current sources, the formulas below should be summarized by the variable *s*. Let us write the third-party radial electric current density in the form [Denisova and Kalinin, 2018]

$$\boldsymbol{j}^{ext} = \frac{I_{s0}(t)}{r^2 sin\theta}\delta_{N_s}(\theta,\theta_s,\varphi,\varphi_s)\left(\vartheta(r-r_{s0}) - \vartheta(r-r_{s1})\right)\boldsymbol{e}_r,$$

wherer_{s1}иr_{s0} are the radial distances corresponding to the positive and negative charges of the lightning generator, with$r_{s0} < r_{s1}; I_{s0}(t)$ being the current strength. The function$\vartheta(r)$ denotes the Heaviside function. The function$\frac{1}{sin\theta}\delta_{N_s}(\theta,\theta_s,\varphi,\varphi_s)$ contains an additional parameterN_s and has the form:

$$\frac{1}{sin\theta}\delta_{N_s}(\theta,\theta_s,\varphi,\varphi_s) = \sum_{n=0}^{N_s}\sum_{k=0}^{n}\frac{\left[Y_{n,k}^{(1)}(\theta,\varphi)Y_{n,k}^{(1)}(\theta_s,\varphi_s) + Y_{n,k}^{(2)}(\theta,\varphi)Y_{n,k}^{(2)}(\theta_s,\varphi_s)\right]}{\left\|Y_{n,k}\right\|^2} =$$

$$= \sum_{n=0}^{N_s}\frac{(2n+1)}{4\pi}P_n(cos\gamma). \quad (6)$$

Зhere is the designation

$$cos\gamma = cos\theta cos\theta_s + sin\theta sin\theta_s\, cos(\varphi - \varphi_s),$$

$Y_{n,k}^{(1)}(\theta,\varphi), Y_{n,k}^{(2)}(\theta,\varphi) -$ real spherical functions; $P_n(cos\gamma) -$ Lejandre polynomials, the norms of spherical functions from the first and second families coincide, so the upper index in writing down the norms $\|Y_{n,k}\|$ is absent.

If $N_s = \infty$, then the series standing in the right part of formula (6) is a decomposition of the Dirac delta function $\frac{1}{sin\theta}\delta(\theta - \theta_s)\delta(\varphi - \varphi_s)$ into a series of spherical functions, which corresponds to the use in problem (1)– (5) of point charges to describe a dipole current source. Then γ is the angle between the radial ray of the observation point and the dipole axis.

If N_s takes a finite value, the sum (6) is a partial sum of the series. Graphs of function (6) are given in [Denisova and Kalinin, 2018]. This function has a maximum at the point , $\theta = \theta_s \varphi = \varphi_s$, is different from zero at all $\theta и \varphi$ and is sign-variable, which makes its physical interpretation difficult. The solution of the problem obtained in Appendix 1 of this paper, as well as the asymptotic formulas of Appendix 2, are valid at any value of N_s including at . $N_s = \infty$

To solve problem (1)– (5) we used the Laplace transform on the time variable [Lavrentiev and Shabat, 1973], applying which we arrive at the boundary value problem for Eq:

$$\left(1 + \frac{p}{4\pi\sigma}\right)\left(\frac{\partial^2 \bar{\Phi}}{\partial r^2} + \frac{2}{r}\frac{\partial \bar{\Phi}}{\partial r} + \frac{1}{r^2}\Delta_{\theta,\varphi}\bar{\Phi}\right) + \frac{1}{H}\frac{\partial \bar{\Phi}}{\partial r} = \frac{1}{\sigma} div \bar{\boldsymbol{j}}^{ext} \qquad (7)$$

with boundary conditions similar to (2)– (4). Through $\bar{\Phi}, \bar{\boldsymbol{j}}^{ext}$ denote the function images Φ and $\boldsymbol{j}^{ext}$. In the ball layer $r_0 < r < r_m$, if $|p| \ll \frac{2\pi\sigma_0 r_0}{H}$, the moduli of the coefficients of Eq. (7) differ little from the moduli of the coefficients of Eq:

$$\left(1 + \frac{p}{4\pi\sigma}\right)\left(\frac{\partial^2 \bar{\Phi}}{\partial r^2} + \frac{1}{r_0^2}\Delta_{\theta,\varphi}\bar{\Phi}\right) + \frac{1}{H}\frac{\partial \Phi}{\partial r} = \frac{1}{\sigma} div \bar{\boldsymbol{j}}^{ext}. \qquad (8)$$

Therefore, the original solution of the boundary value problem for equation (8) with boundary conditions similar to (2)— (4) can be used $t \gg \frac{H}{2\pi\sigma_0 r_0}$ to investigate the electric potential distribution . $\Phi(r, \theta, \varphi, t)$

Among equations (8), we can distinguish the equation corresponding to a stationary dipole oscillator in which the current strength does not change with time $I_{s0}(t) = I_{s0}\vartheta(t)$, but $\bar{I}_{s0}(p) = I_{s0}/p$. In the case of a nonstationary current of the form $I_{s0}(t) = I_{s0}f(t), f(0) = 0$ in the right-hand side of equation (8), we need to replace $\frac{1}{p}$ by the function $\bar{f}(p)$. The solutions of boundary value problems for these equations differ only by a multiplier and the knowledge of the original for the function $\bar{\Phi}$ allows us to write down the solution of the problem with unsteady current as a convolution of functions F and $f'(t)$. In the simplest cases of choosing the type of function $f(t)$, the solution of the problem with unsteady current has a simple form. Let us give some examples.

1. The source stops acting at time $t = T$, modeled by using the function $f(t) = \vartheta(t) - \vartheta(t - T)$. Then the solution of the problem with an unsteady source (denoted by $\Phi^{(\text{нестац})}$) is written in the form:

$$\Phi^{(\text{нестац})}(t) = \Phi(t) - \Phi(t - T)\vartheta(t - T). \tag{9}$$

2. For an instantaneous current source:

$$f(t) = T\delta(t - T), \Phi^{(\text{нестац})}(t) = T\frac{\partial\Phi(t-T)}{\partial t}\vartheta(t - T). \tag{10}$$

3. If $f(t) = 1 - e^{-t/T}$, then $\Phi^{(\text{нестац})} = \frac{1}{T}\int_0^t \Phi\left(t'\right) exp\left(-\frac{(t-t')}{T}\right) dt'. \quad (11)$

These formulas are valid at any fixed point (r, θ, φ) , so the dependence of functions on spatial variables is omitted in the formulas. In all given examples, the solution of the problem for a dipole generator with a nonstationary current is expressed through the solution of the problem with a stationary current.

In this paper, the solution of the boundary value problem for equation (8) with stationary current is found analytically. This allows in the case of a nonstationary current of the form 1— 3 to also write down an analytical solution by substituting the solution with stationary current into formulas (9), (10), (11) in accordance with the physics of the problem. Moreover, the analytical expression of the Green's function of the boundary value problem for equation (8) is obtained in appendix 1. Then, using formula (10), one can write down a solution that differs from the Green's function of the corresponding initial boundary value problem only by a numerical multiplier. Knowledge of the Green's function allows us to represent the solutions of initial boundary value problems for a wide class of unsteady right-hand sides of the equation as a convolution with the Green's function.

The following notations are used in this article: height ,$h = r - r_0 \sigma_m = \sigma(r_m), \sigma_{s0} = \sigma(r_{s0}), \sigma_{s1} = \sigma(r_{s1}), Q_s = \frac{I_{s0} H}{r_0^2}$, $V_{\infty,s} = \frac{Q_s}{4\pi\sigma_0}\left(\frac{\sigma_0}{\sigma_{s0}} - \frac{\sigma_0}{\sigma_{s1}}\right)$.

The detailed analytical solution of the boundary value problem for equation (8) with stationary current and finding the original is given in Appendix 1. To solve the problem, the method of decomposition of the solution by spherical functions is applied; the coefficients of the decomposition depending on the radial variable are expressed through hypergeometric functions. The function (10), in the right part of which the solution (P1.27) found in Appendix 1 with$N_s = \infty$ is substituted, differs from the Green's function only by a multiplier. The Green's function is a generalized function, and in the paper we found its representation in the form of a superposition of two series by Lejandre polynomials.

The stationary model of the current dipole is one of the simplest models of the right-hand side of the equation, for which convergent series in the globular layer are obtained. Formulas (P1.34)− (P1.36) determine the distribution of the electric potential $\Phi(r, \theta, \varphi, t)$ in the whole globular layer$r_0 < r < r_m$ at all$t \gg \frac{H}{2\pi\sigma_0 r_0}$. Функция$\Phi(r, \theta, \varphi, t)$ depends on a large number of parameters:$H, r_0, \theta_s, \varphi_s, h_{s0} = r_{s0} - r_0, h_{s1} = r_{s1} - r_0, h_m = r_m - r_0, N_s$. In the paper we will assume that $h_{s0} = 5$км$, h_{s1} = 10$км$, h_m = 100$км and evaluate the influence on the potential of only the parameter N_s.

1.If$N_s = \infty$, formulas (P1.34)− (P1.36), defining the solution of the boundary value problem, are functional series, which converge in the whole globular layer, except for the points of location of charges, but converge non-uniformly and very slowly. Fig. 1 shows a graph of the stationary part of this solution, normalized to the ionospheric potential, depending on the height of$h$ on the radial beam of the location of charges. To construct the graph, the solution was calculated at several points of the intervals (0, 4.5], [5.5, 9.5], [10.5, 20) of the axis$h$ and linear interpolation was used. In this case, to find the sum of the series to the accuracy of the first two significant digits we have to use partial sums of the series with .$N_s = 30000$

Figure 1.

Numerical investigation of the solution in the nonstationary case, at$N_s = \infty$ by formulas (P1.34)— (P1.35) requires a long calculation and in the paper is carried out only in the upper part of the ball layer at large values of$t$. In appendix 2, at$t \to \infty$ for the potential$\Phi(r, \theta, \varphi, t)$, the asymptotic formula (P2.1), valid at any value ofN_s . is obtained. Formula (P2.1) contains only the summation operation and at$N_s = \infty$ determines the asymptotics of the potential at any point of the globular layer, except for the charge locations. Fig. 2 shows the plots of the stationary solution$\frac{\Phi^{(\text{ст})}}{V_{\infty,s}}$ (dashed line) and the function on the right side of the asymptotic formula (P2.1), also normalized to$V_{\infty,s}$ depending on the variableθ at time$t' = 4\pi\sigma_0 t = 2$ at fixed$h и \varphi$. To find the sum of the series to the nearest tenth, it is sufficient to use .$N_s = 1000$

The left graph corresponds to$h = 70км$, the right— to the upper boundary of the globular layer .$h = 100км$

Figure 2.

The right graphs of Fig. 2 are symmetric with respect to the line$\theta = \pi/2$, which corresponds to condition (2); the maximum values of the functions presented in the figure are equal to 1.37 and 1.45, respectively. Numerical calculations show an insignificant decrease of the potential value in comparison with the ionospheric one at the geomagnetic poles. At$t \to \infty$ in the vicinity of the points (h_m, θ_s, φ_s) and ($h_m, \pi - \theta_s, \varphi_s$)) the nonstationary solution tends to the stationary one from above, and in the vicinity of the points ($h_m, 0, \varphi_s$)и(h_m, π, φ_s)– from below.

The problem of nonuniform convergence of series (P1.34), first of all, is connected with the pointness of charges of the considered current dipole.

3. At any given finite value ofN_s the formulas (P1.34— P1.36) contain only finite sums, which considerably reduces the duration of calculations and excludes discontinuities in the points of charge locations. Moreover, these are not just partial sums of the solution for a current dipole with point charges, but also the solution of the initial boundary value problem with a special right-hand side. Therefore, the graphs given in this paragraph give a qualitative picture of the potential change with time.

Fig. 3

Figure 4.

In the unsteady case, all calculations have been performed for$N_s = 20.$ In Fig. 3 and Fig. 4, the dashed lines show the graph of the stationary solution of the problem and the graphs of the nonstationary solution of the problem (1)− (5), normalized to the ionospheric potential, as a function of the height h on the radial beam of the location of charges at different moments of time in the lower atmosphere. The plots show a monotonic change of the function $\Phi/V_{\infty,s}$ with time in the neighborhood of $h = 5$км , and non-monotonic in the neighborhood of $h = 10$км . Already for $t' = 4\pi\sigma_0 t = 1$ the solution $\Phi/V_{\infty,s}$ differs little from the stationary one $\Phi^{(\text{ст})}/V_{\infty,s}$, with the largest differences in the vicinity of the point $h = 5$км .

4. APPENDIX 1. SOLUTION OF THE INITIAL BOUNDARY VALUE PROBLEM

In the appendix, the solution of the boundary value problem for Eq:

$$\left(1+\frac{p}{4\pi\sigma}\right)\left(\frac{\partial^2\bar{\Phi}_{s0}}{\partial r^2}+\frac{1}{r_0^2}\Delta_{\theta,\varphi}\bar{\Phi}_{s0}\right)+\frac{1}{H}\frac{\partial\bar{\Phi}_{s0}}{\partial r}=$$

$$=\frac{Q_s}{\sigma p H sin\theta}\delta_{N_s}(\theta,\theta_s,\varphi,\varphi_s)\delta(r-r_{s0}) \qquad (\Pi1.1)$$

with boundary conditions similar to (2)− (4) ($Q_s = \frac{I_{s0}H}{r_0^2}$). The solution of the boundary value problem for equation (8) is written in the form:

$$\bar{\Phi}=\bar{\Phi}_{s0}-\bar{\Phi}_{s1}. \qquad (\Pi1.2)$$

If $N_s = \infty$, the solution of equation (P1.1) with conditions (2)− (4) differs from the Green's function only by a multiplier.

In equation (P1.1) and boundary conditions similar to (2)− (4), let us substitute variables:

$$\begin{cases}\sigma=\sigma_0 exp\left(\frac{r-r_0}{H}\right)\\ \mu=cos\theta\end{cases}$$

and formulate the boundary value problem in the domain : $\sigma_0 < \sigma < \sigma_m$

$$\sigma^2\frac{\partial^2\bar{\Phi}_{s0}}{\partial\sigma^2}+\sigma\frac{(8\pi\sigma+p)}{(4\pi\sigma+p)}\frac{\partial\bar{\Phi}_{s0}}{\partial\sigma}+\frac{H^2}{r_0^2}\Delta_{\mu,\varphi}\bar{\Phi}_{s0}=$$

$$= \frac{4\pi\sigma Q_s}{p(4\pi\sigma + p)} \delta(\sigma - \sigma_{s0}) \delta_{N_s}(\mu, \mu_s, \varphi, \varphi_s), \tag{П1.3}$$

$$\bar{\Phi}_{s0}(\sigma_m, \mu, \varphi, p) = \bar{\Phi}_{s0}(\sigma_m, -\mu, \varphi, p), \tag{П1.4}$$

$$\frac{\partial \bar{\Phi}_{s0}(\sigma_m, \mu, \varphi, p)}{\partial \sigma} = -\frac{\partial \bar{\Phi}_{s0}(\sigma_m, -\mu, \varphi, p)}{\partial \sigma}, \tag{П1.5}$$

$$\bar{\Phi}_{s0}(\sigma_0, \mu, \varphi, p) = 0. \tag{П1.6}$$

Here. $\sigma_m = \sigma(r_m), \mu_s = cos\theta_s$.

The solution of the problem is represented as a series of spherical functions

$$\bar{\Phi}_{s0} = \bar{A}_{00}(\sigma, p) + \sum_{n=1}^{\infty} \sum_{k=0}^{n} \left(\bar{A}_{nk}(\sigma, p) Y_{n,k}^{(1)}(\mu, \varphi) + \bar{B}_{nk}(\sigma, p) Y_{n,k}^{(2)}(\mu, \varphi) \right). \tag{П1.7}$$

The function represented by the series (P1.7) is a solution of equation (P1.3) if the coefficients of$\bar{A}_{nk}(\sigma, p)$ satisfy the inhomogeneous equation:

$$\sigma^2 \frac{\partial^2 \bar{A}_{nk}}{\partial \sigma^2} + \sigma \frac{(8\pi\sigma + p)}{(4\pi\sigma + p)} \frac{\partial \bar{A}_{nk}}{\partial \sigma} - \frac{n(n+1)H^2}{r_0^2} \bar{A}_{nk} =$$

$$= \frac{4\pi\sigma Q_s}{p(4\pi\sigma + p)} \frac{Y_{nk}^{(1)}(\mu_s, \varphi_s)}{\|Y_{nk}\|^2} \delta(\sigma - \sigma_{s0}), 0 \le n \le N_s, \tag{П1.8}$$

for$0 \le n \le N_s$ and the corresponding homogeneous equation if$n > N_s$. From condition (P1.6) follows the condition

$$\bar{A}_{nk}(\sigma_0, p) = 0. \tag{П1.9}$$

The conditions (P1.4— P1.5), which take into account the coupling of electric fields in magnetically conjugate points at the upper boundary of the atmosphere, correspond to the conditions (P1.4 P1.5):

$$\bar{A}_{nk}(\sigma_m, p) = 0, \text{если} n + k = \text{an odd number}, \tag{П1.10}$$

$$\frac{\partial \bar{A}_{nk}}{\partial \sigma}(\sigma_m, p) = 0, \text{если} n + k = \text{an even number}. \qquad (\Pi1.11)$$

Similar boundary value problems are obtained for the functions $\bar{B}_{nk}(\sigma, p)$, only in the right part of equation (P1.8) there will be a spherical function with upper index 2. For $n > N_s$, due to the homogeneity of the equation and homogeneity of the boundary conditions, all coefficients of $\bar{A}_{nk}(\sigma, p)$ and $\bar{B}_{nk}(\sigma, p)$ are equal to zero.

If $\left|\frac{p}{4\pi\sigma}\right| < 1$, the homogeneous equation corresponding to (P1.8) has two linearly independent solutions, which are expressed through hypergeometric functions [Gradstein and Ryzhik, 1963]:

$$\bar{A}_{nk}^{(1,\text{одн})}(\sigma, p) = \left(-\frac{p}{4\pi\sigma}\right)^{\frac{1+\xi_n}{2}} F\left(\alpha_n, \beta_n, \alpha_n + \beta_n, -\frac{p}{4\pi\sigma}\right),$$

$$\bar{A}_{nk}^{(2,\text{одн})}(\sigma, p) = \left(-\frac{p}{4\pi\sigma}\right)^{\frac{1-\xi_n}{2}} F\left(1 - \alpha_n, 1 - \beta_n, 2 - \alpha_n - \beta_n, -\frac{p}{4\pi\sigma}\right). \qquad (\Pi1.12)$$

Here.

$$\alpha_n = \frac{1}{2}\left(1 + \xi_n - \sqrt{\xi_n^2 - 1}\right), \beta_n = \frac{1}{2}\left(1 + \xi_n + \sqrt{\xi_n^2 - 1}\right), \qquad (\Pi1.12a)$$

where $\xi_n = \sqrt{1 + \frac{4n(n+1)H^2}{r_0^2}}$. In finding the solution of the inhomogeneous equation (P1.8) with conditions (P1.9-P1.11), we can use the last formula of paragraph 24.2 of the reference [Kamke, 1976] and then the properties of the delta function. The formula is obtained by the method of variation of arbitrary constants and contains the vronskian of solutions of the homogeneous equation. To compute the vronskian of functions (P1.12), derivatives from hypergeometric functions are used. To shorten the notes of the hypergeometric functions used in this paper, we introduce the notations:

$$F_n^{(1)}(x) = F(\alpha_n, \beta_n, \alpha_n + \beta_n, x),$$

$$F_n^{(2)}(x) = F(1 - \alpha_n, 1 - \beta_n, 2 - \alpha_n - \beta_n, x),$$

$$F_n^{(3)}(x) = F(\alpha_n + 1, \beta_n + 1, \alpha_n + \beta_n + 1, x),$$

$$F_n^{(4)}(x) = F(2 - \alpha_n, 2 - \beta_n, 3 - \alpha_n - \beta_n, x).$$

Functions numbered 3, 4 appear when differentiating$F_n^{(1)}, F_n^{(2)}$ on the independent variable .x

Depending on what is the number$n+k$, even or odd, for solutions of boundary value problems (P1.8) − (P1.11) we obtain:

$$\bar{A}_{nk}(\sigma,p) = \begin{cases} \dfrac{Y_{nk}^{(1)}(\mu_s,\varphi_s)}{\|Y_{nk}\|^2}\bar{\tilde{R}}_n(\sigma,p), \text{если} n+k - \text{an even number} \\ \dfrac{Y_{nk}^{(1)}(\mu_s,\varphi_s)}{\|Y_{nk}\|^2}\bar{\tilde{\tilde{R}}}_n(\sigma,p), \text{если} n+k - \text{an odd number} \end{cases}. \quad (\Pi1.13)$$

The function$\bar{\tilde{R}}_n(\sigma,\sigma_{s0},p)$ is written using the formulas

$$\bar{\tilde{R}}_n(\sigma,\sigma_{s0},p) =$$

$$= \begin{cases} \dfrac{Q_s}{p\xi_n\sqrt{\sigma\sigma_{s0}}}\left(\dfrac{\sigma}{\sigma_{s0}}\right)^{\frac{\xi_n}{2}} \dfrac{\tilde{V}_n\left(-\frac{p}{4\pi\sigma_{s0}},-\frac{p}{4\pi\sigma_m}\right)I_n\left(-\frac{p}{4\pi\sigma_0},-\frac{p}{4\pi\sigma}\right)}{\tilde{V}_n\left(-\frac{p}{4\pi\sigma_0},-\frac{p}{4\pi\sigma_m}\right)}, \sigma_0<\sigma<\sigma_{s0} \\ \dfrac{Q_s}{p\xi_n\sqrt{\sigma\sigma_{s0}}}\left(\dfrac{\sigma_{s0}}{\sigma}\right)^{\frac{\xi_n}{2}} \dfrac{\tilde{V}_n\left(-\frac{p}{4\pi\sigma},-\frac{p}{4\pi\sigma_m}\right)I_n\left(-\frac{p}{4\pi\sigma_0},-\frac{p}{4\pi\sigma_{s0}}\right)}{\tilde{V}_n\left(-\frac{p}{4\pi\sigma_0},-\frac{p}{4\pi\sigma_m}\right)}, \sigma_{s0}<\sigma<\sigma_m \end{cases}, \quad (\Pi1.14)$$

where

$$I_n(x_1,x_2) = \left(\frac{x_2}{x_1}\right)^{\xi_n} F_n^{(1)}(x_2)F_n^{(2)}(x_1) - F_n^{(1)}(x_1)F_n^{(2)}(x_2), \quad (\Pi1.15)$$

$$\tilde{V}_n(x_1,x_2) = \left(\frac{x_2}{x_1}\right)^{\xi_n} \tilde{\phi}_n^{(1)}(x_2)F_n^{(2)}(x_1) + F_n^{(1)}(x_1)\tilde{\phi}_n^{(2)}(x_2). \quad (\Pi1.16)$$

Here.

$$\tilde{\phi}_n^{(2)}(x) = \frac{\xi_n-1}{\xi_n+1}F_n^{(2)}(x) - \frac{x}{(1+\xi_n)}F_n^{(4)}(x) \quad (\Pi1.17)$$

$$\tilde{\phi}_n^{(1)}(x) = F_n^{(1)}(x) + \frac{x}{(1+\xi_n)}F_n^{(3)}(x). \quad (\Pi1.18)$$

For the function $\bar{\tilde{\tilde{R}}}_n(\sigma,\sigma_{s0},p)$ we obtain an expression similar to (P1.14— P1.16), only in these formulas two waves should be put over the functions having one wave in the notations, and it should be taken into account that the formulas defining the functions $\tilde{\tilde{\phi}}_n^{(2)}(x), \tilde{\tilde{\phi}}_n^{(1)}(x)$, are different:

$$\tilde{\tilde{\phi}}_n^{(2)}(x) = -F_n^{(2)}(x), \tilde{\tilde{\phi}}_n^{(1)}(x) = F_n^{(1)}(x). \qquad (\Pi1.19)$$

The coefficient $\bar{B}_{nk}(\sigma,p)$ is of the form (P1.13), only a spherical function with upper index 2 is used as a multiplier in front of $\bar{\tilde{R}}_n(\sigma,p)$.

Since the function (P1.14) is symmetric with respect to the variables σ,σ_{s0} , it is convenient to introduce the parameter σ_{s0} into the arguments of the functions $\bar{\Phi}_{s0}, \bar{A}_{nk}, \bar{B}_{nk}, \bar{\tilde{R}}_n, \bar{\tilde{\tilde{R}}}_n$ and to use notations in which the order of these variables is important, for example,

$$\bar{\Phi}_{s0} = \begin{cases} \bar{\Phi}(\sigma,\sigma_{s0},\mu,\varphi,p), \text{если} \sigma_0 < \sigma < \sigma_{s0} \\ \bar{\Phi}(\sigma_{s0},\sigma,\mu,\varphi,p), \text{если} \sigma_{s0} < \sigma < \sigma_m \end{cases}$$

and consider only the case $\sigma_0 < \sigma < \sigma_{s0}$. If in the solution found for $\sigma_0 < \sigma < \sigma_{s0}$, we replace the first argument σ на σ_{s0} и второй аргумент σ_{s0} на σ , we obtain a solution in the domain .$\sigma_{s0} < \sigma < \sigma_m$

After substituting $\bar{A}_{nk}(\sigma,\sigma_{s0},p), \bar{B}_{nk}(\sigma,\sigma_{s0},p)$ into the series (P1.7), the inner sum over k , at each fixed n, , let us split it into two sums depending on the number $n+k$, even or odd [Denisova and Kalinin, 2018]. Then, using the addition theorem of the connected Lejandre functions, we will have

$$\bar{\Phi} = \bar{A}_{00}(\sigma,\sigma_{s0},p) + \sum_{n=1}^{N_s} \frac{(2n+1)}{8\pi} \{ \left(\bar{\tilde{R}}_n(\sigma,\sigma_{s0},p) + \bar{\tilde{\tilde{R}}}_n(\sigma,\sigma_{s0},p) \right) P_n(cos\gamma) +$$

$$+\left(\bar{\tilde{R}}_n(\sigma,\sigma_{s0},p)-\bar{\tilde{\tilde{R}}}_n(\sigma,\sigma_{s0},p)\right)P_n(cos\gamma_1)\}. \qquad (\Pi 1.20)$$

Here.

$$cos\gamma_1=-\mu\mu_s+\sqrt{1-\mu^2}\sqrt{1-\mu_s^2}cos(\varphi-\varphi_s),$$

where γ_1 is the angle between the radial ray directed to the observation point and the radial ray containing the points conjugate to the dipole charge locations.

Figure 5.

To find the inverse Laplace transform of the function $\bar{\Phi}(\sigma,\sigma_{s0},\mu,\varphi,p)$ we need to find the inverse transform of the functions $\bar{A}_{00}(\sigma,\sigma_{s0},p),\bar{\tilde{R}}_n(\sigma,\sigma_{s0},p),\bar{\tilde{\tilde{R}}}_n(\sigma,\sigma_{s0},p)$. Let us first consider the function $\bar{\tilde{R}}_n(\sigma,\sigma_{s0},p)$. This function is defined by formula (P1.14) if $\left|\frac{p}{4\pi\sigma_0}\right|<1,$ and can be continued to the complex plane $p=p'+ip''$, since all hypergeometric functions included in formula (P1.14) can be continued to the complex plane. Given the special points of these hypergeometric functions, we will use the closed contour C in the complex plane $p=p'+ip''$, shown in Fig. 5. The contour circumscribes the special points of the function $\bar{\tilde{R}}_n(\sigma,\sigma_{s0},p)$. These points are located on the negative part of the real axis p' and have coordinates: $-4\pi\sigma_m,-4\pi\sigma_{s0},-4\pi\sigma,-4\pi\sigma_0$. Inside the contour C, the integrand has a first-order pole at $p=0$, therefore

$$\frac{1}{2\pi i}\int_C \bar{\tilde{R}}_n(\sigma,\sigma_{s0},p)exp\,(pt)dp=\underset{p=0}{res}\left(\bar{\tilde{R}}_n(\sigma,\sigma_{s0},p)exp\,(pt)\right). \qquad (\Pi 1.21)$$

Let us denote this deduction by $\tilde{R}_n^{(стац))}(\sigma,\sigma_{s0})$. Given that the hypergeometric functions at $p=0$ are equal to 1, we obtain

$$\tilde{R}_n^{(\text{стац})}(\sigma,\sigma_{s0}) = \frac{Q_s}{\xi_n\sqrt{\sigma\sigma_{s0}}}\left(\frac{\sigma}{\sigma_{s0}}\right)^{\frac{\xi_n}{2}} \frac{\left(\left(\frac{\sigma_{s0}}{\sigma_m}\right)^{\xi_n} + \frac{\xi_n - 1}{\xi_n + 1}\right)\left(\left(\frac{\sigma_0}{\sigma}\right)^{\xi_n} - 1\right)}{\left(\left(\frac{\sigma_0}{\sigma_m}\right)^{\xi_n} + \frac{\xi_n - 1}{\xi_n + 1}\right)}. \quad (\Pi 1.22)$$

For the continuation of the hypergeometric functions $F_n^{(i)}(z)$ over the exterior of the unit circle of the complex plane z with a cut along the real axis from 1 to ∞ we used the formula [Gradshtein and Ryzhik, 1963, f. 9.132(2)]. According to this formula, the function $\bar{\tilde{R}}_n(\sigma,\sigma_{s0},p)$ on the upper and lower banks of the section along the negative part of the real axis takes complex-conjugate values. The contour of Fig. 5 is constructed taking into account the complex arguments of the hypergeometric functions defining $\bar{\tilde{R}}_n(\sigma,\sigma_{s0},p)$. Setting the radius of the large circle in formula (P1.21) to ∞ , and the radii of the small semicircles to zero, we obtain

$$\tilde{R}_n(\sigma,\sigma_{s0},t) = \tilde{R}_n^{(\text{стац})}(\sigma,\sigma_{s0}) - \frac{1}{\pi}\int_{-4\pi\sigma_m}^{-4\pi\sigma_0} Im\bar{\tilde{R}}_n(\sigma,\sigma_{s0},p')\, exp(p't)\, dp'. \quad (\Pi 1.23)$$

Since the formulas for the continuation of hypergeometric functions $F_n^{(1)}(z), F_n^{(2)}(z)$ beyond the unit circle [Gradstein and Ryzhik, 1963, f. 9.132(2)] were used to find $Im\bar{\tilde{R}}_n$ on the upper bank of the section at $-4\pi\sigma_m < p' < -4\pi\sigma_0$, two more hypergeometric functions appear:

$$F_n^{(5)}(x) = F(\alpha_n, 1-\beta_n, 1+\alpha_n-\beta_n, x),$$

$$F_n^{(6)}(x) = F(1-\alpha_n, \beta_n, 1-\alpha_n+\beta_n, x).$$

In the integral we replace the variable of integration: $\eta = -\frac{p'}{4\pi\sigma_0}$, select the constant multiplier from the function $Im\bar{\tilde{R}}_n(\sigma,\sigma_{s0},-4\pi\sigma_0\eta)$ and introduce a new function

$$Im\bar{\tilde{R}}_n(\sigma,\sigma_{s0},-4\pi\sigma_0\eta) = -\frac{Q_s}{4\sigma_0^2}\tilde{C}_n(\sigma,\sigma_{s0},\eta).$$

As a result, let us rewrite the function $\tilde{R}_n(\sigma,\sigma_{s0},t)$ in the form

$$\tilde{R}_n(\sigma,\sigma_{s0},t) = \tilde{R}_n^{(\text{стац}))}(\sigma,\sigma_{s0}) + \frac{Q_s}{\sigma_0}\int_1^{\sigma_m/\sigma_0} \tilde{C}_n(\sigma,\sigma_{s0},\eta)\,exp(-4\pi\sigma_0 t\eta)\,d\eta. \quad (\Pi1.24)$$

Considering the special points of the function $\tilde{R}_n$, , we divide the integration interval $\left(1, {}^{\sigma_m}/_{\sigma_0}\right)$ in formula (P1.24) into three parts, in each of which we obtain a different analytical expression for the function $\tilde{C}_n(\sigma,\sigma_{s0},\eta)$. We will use the following notations:

$$\tilde{C}_n(\sigma,\sigma_{s0},\eta) = \begin{cases} \tilde{C}_n^{(1)}, \eta \in \left(1, {}^{\sigma}/_{\sigma_0}\right) \\ \tilde{C}_n^{(2)}, \eta \in \left({}^{\sigma}/_{\sigma_0}, {}^{\sigma_{s0}}/_{\sigma_0}\right) \\ \tilde{C}_n^{(3)}, \eta \in \left({}^{\sigma_{s0}}/_{\sigma_0}, {}^{\sigma_m}/_{\sigma_0}\right) \end{cases}, \quad (\Pi1.25)$$

where

$$\tilde{C}_n^{(1)}(\sigma,\sigma_{s0},\eta) = \frac{\left(\frac{\sigma_0}{\sqrt{\sigma\sigma_{s0}}}\right)^{\xi_n+1}\eta^{\xi_n-1}L_n^2\left(\frac{1}{\eta}\right)\tilde{V}_n\left(\frac{\sigma_0}{\sigma}\eta,\frac{\sigma_0}{\sigma_m}\eta\right)\tilde{V}_n\left(\frac{\sigma_0}{\sigma_{s0}}\eta,\frac{\sigma_0}{\sigma_m}\eta\right)}{\left|\tilde{V}_n\left(\eta,\frac{\sigma_0}{\sigma_m}\eta\right)\right|^2}$$

$$\tilde{C}_n^{(2)}(\sigma,\sigma_{s0},\eta) = -\frac{\left(\frac{\sqrt{\sigma}}{\sqrt{\sigma_{s0}}}\right)^{\xi_n+1}L_n\left(\frac{1}{\eta}\right)M_n\left(\frac{1}{\eta},\frac{\sigma}{\sigma_0\eta}\right)\tilde{V}_n\left(\frac{\sigma_0}{\sigma_{s0}}\eta,\frac{\sigma_0}{\sigma_m}\eta\right)\tilde{W}_n\left(\frac{\sigma_0}{\sigma_m}\eta\right)}{\eta^2\left|\tilde{V}_n\left(\eta,\frac{\sigma_0}{\sigma_m}\eta\right)\right|^2}$$

$$\tilde{C}_n^{(3)}(\sigma,\sigma_{s0},\eta) = \frac{\left(\frac{\sqrt{\sigma\sigma_{s0}}}{\sigma_0}\right)^{\xi_n+1}M_n\left(\frac{1}{\eta},\frac{\sigma_{s0}}{\sigma_0\eta}\right)M_n\left(\frac{1}{\eta},\frac{\sigma}{\sigma_0\eta}\right)\tilde{W}_n^2\left(\frac{\sigma_0}{\sigma_m}\eta\right)}{\eta^{\xi_n+3}\left|\tilde{V}_n\left(\eta,\frac{\sigma_0}{\sigma_m}\eta\right)\right|^2}.$$

Here.

$$M_n(x_1,x_2) = \frac{1}{\beta_n-\alpha_n}\left(\left(\frac{x_1}{x_2}\right)^{\alpha_n}F_n^{(3)}(x_1)F_n^{(4)}(x_2) - \left(\frac{x_1}{x_2}\right)^{\beta_n}F_n^{(3)}(x_2)F_n^{(4)}(x_1)\right),$$

$$L_n(x) = \frac{\delta_n^{(1)}\Gamma(1-\alpha_n+\beta_n)}{\Gamma(2-\alpha_n)\Gamma(\beta_n)}x^{\alpha_n}F_n^{(3)}(x) - \frac{\delta_n^{(2)}\Gamma(1+\alpha_n-\beta_n)}{\Gamma(\alpha_n)\Gamma(2-\beta_n)}x^{\beta_n}F_n^{(4)}(x),$$

$$\tilde{W}_n(x) = \frac{\Gamma(3-\alpha_n-\beta_n)}{2\Gamma(2-\alpha_n)\Gamma(2-\beta_n)}x^{\xi_n}\tilde{\Phi}_n^{(1)}(x) + \frac{\Gamma(\alpha_n+\beta_n)}{\Gamma(\alpha_n)\Gamma(\beta_n)}\tilde{\Phi}_n^{(2)}(x),$$

$$\delta_n^{(1,2)} = \frac{1}{2}\left(\pm 1 - \sqrt{\frac{\xi_n - 1}{\xi_n + 1}}\right).$$

The function $\tilde{\tilde{R}}_n(\sigma, \sigma_{s0}, t)$ is represented by the formula similar to (P1.24), only all letters with one wave should be replaced by letters with two waves. In this case, the function $\tilde{\tilde{R}}_n^{(стац))}$ has the form:

$$\tilde{\tilde{R}}_n^{(стац))}(\sigma, \sigma_{s0}) = \frac{Q_s}{\xi_n\sqrt{\sigma\sigma_{s0}}}\left(\frac{\sigma}{\sigma_{s0}}\right)^{\frac{\xi_n}{2}} \frac{\left(\left(\frac{\sigma_{s0}}{\sigma_m}\right)^{\xi_n} - 1\right)\left(\left(\frac{\sigma_0}{\sigma}\right)^{\xi_n} - 1\right)}{\left(\left(\frac{\sigma_0}{\sigma_m}\right)^{\xi_n} - 1\right)}. \quad (П1.26)$$

To find the coefficient $A_{00}(\sigma, \sigma_{s0}, p)$ no special functions are required and returning to the original we obtain

$$A_{00}(\sigma, \sigma_{s0}, t) = \frac{Q_s}{4\pi}\left(\frac{1}{\sigma} - \frac{1}{\sigma_0}\right) + \frac{Q_s}{4\pi\sigma_0}\int_1^{\sigma/\sigma_0} \frac{1}{\eta^2} e^{-4\pi\sigma_0 t\eta} d\eta. \ \sigma_0 < \sigma < \sigma_{s0}$$

Considering (P1.24), after regrouping the summands, we write the solution (P1.20) in the form

$$\Phi(\sigma, \sigma_{s0}, \mu, \varphi, t) = \Phi^{(стац)}(\sigma, \sigma_{s0}, \mu, \varphi) + \frac{Q_s}{4\pi\sigma_0}\int_1^{\frac{\sigma}{\sigma_0}} \frac{1}{\eta^2} e^{-4\pi\sigma_0 t\eta} d\eta +$$

$$+\sum_{n=1}^{N_s}(2n+1)\left(P_n(cos\gamma)B_n(\sigma, \sigma_{s0}, t) + P_n(cos\gamma_1)D_n(\sigma, \sigma_{s0}, t)\right). \quad (П1.27)$$

Here.

$$\Phi^{(стац)}(\sigma, \sigma_{s0}, \mu, \varphi) = \frac{Q_s}{4\pi}\left(\frac{1}{\sigma} - \frac{1}{\sigma_0}\right) + \sum_{n=1}^{N_s}\frac{(2n+1)}{4\pi}\{R_n^{(стац)}(\sigma, \sigma_{s0})P_n(cos\gamma) +$$

$$+T_n^{(стац)}(\sigma, \sigma_{s0})P_n(cos\gamma_1)\} \quad (П1.28)$$

$$R_n^{(стац)}(\sigma, \sigma_{s0}) = \frac{1}{2}\left\{\tilde{R}_n^{(стац)}(\sigma, \sigma_{s0}) + \tilde{\tilde{R}}_n^{(стац)}(\sigma, \sigma_{s0})\right\} \quad (П1.29)$$

$$T_n^{(стац)}(\sigma,\sigma_{s0}) = \frac{1}{2}\left\{\tilde{R}_n^{(стац)}(\sigma,\sigma_{s0}) - \tilde{\tilde{R}}_n^{(стац)}(\sigma,\sigma_{s0})\right\} (\Pi 1.30)$$

$$\bar{B}_n(\sigma,\sigma_{s0},\eta) = \frac{Q_s}{8\pi\sigma_0}\left\{\tilde{C}_n(\sigma,\sigma_{s0},\eta) + \tilde{\tilde{C}}_n(\sigma,\sigma_{s0},\eta)\right\} (\Pi 1.31)$$

$$\bar{D}_n(\sigma,\sigma_{s0},\eta) = \frac{Q_s}{8\pi\sigma_0}\left\{\tilde{C}_n(\sigma,\sigma_{s0},\eta) - \tilde{\tilde{C}}_n(\sigma,\sigma_{s0},\eta)\right\} (\Pi 1.32)$$

$$B_n(\sigma,\sigma_{s0},t) = \int_1^{\frac{\sigma_m}{\sigma_0}} \bar{B}_n(\sigma,\sigma_{s0},\eta)e^{-4\pi\sigma_0 t\eta}d\eta (\Pi 1.33)$$

$$D_n(\sigma,\sigma_{s0},t) = \int_1^{\frac{\sigma_m}{\sigma_0}} \bar{D}_n(\sigma,\sigma_{s0},\eta)e^{-4\pi\sigma_0 t\eta}d\eta. (\Pi 1.34)$$

All functions (P1.27)– (P1.33) are written for the case$\sigma_0 < \sigma < \sigma_{s0}$. If$\sigma_{s0} < \sigma < \sigma_m$ in the right part of formulas (P1.27)– (P1.33) should be interchanged .$\sigma и \sigma_{s0}$

If$N_s = \infty$, equation (P1.1) differs from the equation for the Green's function only by a multiplier. Therefore, substituting the solution of (P1.27) into formula (10), we obtain, with multiplier accuracy, the Green's function of the initial boundary value problem for the equation corresponding to (8).

Below we write down the solution of the problem for a stationary current dipole. Replacing in formulas (P1.27)– (P1.34)σ_{s0} byσ_{s1} , according to formula (P1.2) we obtain:

$$\Phi_s(\sigma,\mu,\varphi,t) = \Phi^{(стац)}(\sigma,\sigma_{s0},\mu,\varphi) - \Phi^{(стац)}(\sigma,\sigma_{s1},\mu,\varphi) +$$

$$\sum_{n=1}^{N_s}(2n+1)\{P_n(cos\gamma)\big(B_n(\sigma,\sigma_{s0},t) - B_n(\sigma,\sigma_{s1},t)\big) +$$

$$+P_n(cos\gamma_1)(D_n(\sigma,\sigma_{s0},t) - D_n(\sigma,\sigma_{s1},t)\}(\Pi 1.34)$$

in the field$\sigma_0 < \sigma < \sigma_{s0}$.

In the$\sigma_{s0} < \sigma < \sigma_{s1}$ region, the time contribution is also given by the summand corresponding to$n = 0$,

$$\Phi_s(\sigma,\mu,\varphi,t) = \Phi^{(\text{стац})}(\sigma_{s0},\sigma,\mu,\varphi) - \Phi^{(\text{стац})}(\sigma,\sigma_{s1},\mu,\varphi) - \frac{Q_s}{4\pi\sigma_0}\int_{\frac{\sigma_{s0}}{\sigma_0}}^{\frac{\sigma}{\sigma_0}} \frac{1}{\eta^2} e^{-4\pi\sigma_0 t\eta} d\eta +$$

$$+\sum_{n=1}^{N_s}(2n+1)\left\{P_n(cos\gamma)\big(B_n(\sigma_{s0},\sigma,t) - B_n(\sigma,\sigma_{s1},t)\big) +\right.$$

$$\left. +P_n(cos\gamma_1)\big(D_n(\sigma_{s0},\sigma,t) - D_n(\sigma,\sigma_{s1},t)\big)\right\}. \quad (\Pi1.35)$$

In the area$\sigma_{s1} < \sigma < \sigma_m$ we get

$$\Phi_s(\sigma,\mu,\varphi,t) = \Phi^{(\text{стац})}(\sigma_{s0},\sigma,\mu,\varphi) - \Phi^{(\text{стац})}(\sigma_{s1},\sigma,\mu,\varphi) - \frac{Q_s}{4\pi\sigma_0}\int_{\frac{\sigma_{s0}}{\sigma_0}}^{\frac{\sigma_{s1}}{\sigma_0}} \frac{1}{\eta^2} e^{-4\pi\sigma_0 t\eta} d\eta +$$

$$+\sum_{n=1}^{N_s}(2n+1)\left\{P_n(cos\gamma)\big(B_n(\sigma_{s0},\sigma,t) - B_n(\sigma_{s1},\sigma,t)\big) +\right.$$

$$\left. +P_n(cos\gamma_1)\big(D_n(\sigma_{s0},\sigma,t) - D_n(\sigma_{s1},\sigma,t)\big)\right\}. \quad (\Pi1.36)$$

5. APPENDIX 2. ASYMPTOTICS OF THE SOLUTION OF THE PROBLEM AT$t \to \infty$

Asymptotic formulas at$t \to \infty$ for solutions (P1.34– P1.36) can be found using the Laplace method. The determining role here is played by the behavior of integrals' integrand functions (P1.32, P1.33) in the neighborhood of the point$\eta = 1$. Since the function$\overset{\sim(1)}{C_n}(\sigma,\sigma_{s0},\eta)$ is symmetric on the variables$\sigma и \sigma_{s0}$, the main asymptotic term in all formulas (P1.34, P1.35, P1.36) is the same, and the asymptotic formula has the form:

$$\Phi_s(\sigma,\mu,\varphi,t) \approx \Phi_s^{(\text{стац})}(\sigma,\mu,\varphi) + \frac{exp(-4\pi\sigma_0 t)}{4\pi\sigma_0 t ln^2(4\pi\sigma_0 t)} f_s(\sigma,\mu,\varphi), 4\pi\sigma_0 t \gg 1, t \to \infty. \quad (\Pi2.1)$$

Here.

$$f_s(\sigma,\mu,\varphi) = f(\sigma,\sigma_{s0},\mu,\varphi) - f(\sigma,\sigma_{s1},\mu,\varphi)$$

$$f(\sigma,\sigma_{s0},\mu,\varphi) = \frac{Q}{4\pi\sigma_0}\sum_{n=1}^{N_s}(2n+1)\{a_n(\sigma,\sigma_{s0})P_n\,(cos\gamma) + b_n(\sigma,\sigma_{s0})P_n(cos\gamma_1)\}$$

$$a_n(\sigma,\sigma_{s0}) = \frac{1}{2}\left(\tilde{a}_n(\sigma,\sigma_{s0}) + \tilde{\tilde{a}}_n(\sigma,\sigma_{s0})\right)$$

$$b_n(\sigma,\sigma_{s0}) = \frac{1}{2}\left(\tilde{a}_n(\sigma,\sigma_{s0}) - \tilde{\tilde{a}}_n(\sigma,\sigma_{s0})\right)$$

$$\tilde{a}_n(\sigma,\sigma_{s0}) = \frac{\left(\frac{\sigma_0}{\sqrt{\sigma\sigma_{s0}}}\right)^{\xi_n+1}\tilde{V}_n\left(\frac{\sigma_0}{\sigma},\frac{\sigma_0}{\sigma_m}\right)\tilde{V}_n\left(\frac{\sigma_0}{\sigma_{s0}},\frac{\sigma_0}{\sigma_m}\right)}{\tilde{W}_n^2\left(\frac{\sigma_0}{\sigma_m}\right)}. \quad (\Pi 2.2)$$

Note that the asymptotic formula for the spherical mean from the potential is not obtained from formula (P2.1). This is due to the fact that formulas (P1.35)— (P1.36) contain an integral independent ofθ andφ . It is this integral that determines the asymptotic formula for the spherical mean potential. For example, in the region $h_{s1} < h < h_m$

$$\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \Phi_s sin\theta d\theta d\varphi \approx V_{\infty,s} - \frac{Q_s}{4\pi\sigma_{s0}}\frac{exp(-4\pi\sigma_{s0}t)}{4\pi\sigma_{s0}t}, t \to \infty. \quad (\Pi 2.3)$$

The time multiplier of formula (P2.1) depends onσ_0 , and of formula (P2.3) onσ_{s0} . The spherical mean approaches at$t \to \infty$ to$V_{\infty,s}$ from below. Numerical calculations using formula (P2.1) (see Fig. 2) show that at the upper boundary of the atmosphere in the vicinity of the points (θ_s,φ_s) and $(\pi-\theta_s,\varphi_s)$) the potentialΦ_s tends to$\Phi_s^{(\text{стац})}$ from above, and in the vicinity of the points $(0,\varphi_s)$, and (π,φ_s) — снизу.

The results obtained in this paper can be used in modeling the global electric circuit taking into account the influence of the magnetosphere on the electric field distribution in the atmosphere.

6. KEY FINDINGS

1. In this paper, an analytical solution of the nonstationary problem for the electric field potential of a current dipole in the atmosphere occupying a globular layer, the conductivity of

which increases exponentially along the radius, with boundary conditions that take into account the coupling of the electric potential and current at magnetically conjugate points of the upper boundary of the globular layer is found. The analytical solution is represented by formulas (P1.34)– (P1.36) for a stationary current dipole and (9)– (11) for the simplest cases of unsteady current.

2. An analytical expression for the Green's function of the initial boundary value problem, for the equation corresponding to equation (8), is obtained. (Formulas (10), (P1.27)).

3. Numerical analysis of the electric field potential change with time for the model right parts of the equation ($N_s = 20$) on the radial ray of the location of charges in the lower atmosphere is carried out. The monotonic tendency of the electric field potential with time at $t \to \infty$ to the stationary potential in the vicinity of the negative charge of the thunderstorm cloud and non-monotonic in the vicinity of the positive charge is shown.

4. Asymptotic formulas (P2.1)– (P2.2) for the electric potential of the current dipole at $t \to \infty$, taking into account the dependence on spatial coordinates, are obtained. The variation of the electric field potential with time in the upper part of the globular layer has been analyzed for the current dipole ($N_s = \infty$) . It is shown that on the axis of the current dipole location at the upper boundary of the atmosphere the electric field potential decreases with time, while at the geomagnetic poles it increases.

5. In the study of more complex model problems with a distributed current source, the results obtained in the paper may be useful, since they allow us to write down an analytical solution for a wide class of right-hand sides of equation (8) in the form of convolution with the Green's function.

REFERENCE

– *Gradstein I. S., Ryzhik I. M.* Tables of integrals, sums, series and products. Moscow: Fizmatgiz. 1100 c. 1963.

– *Denisova N. A., Kalinin A. B.* Influence of the choice of boundary conditions on the electric field distribution in the global electric circuit models // Izv. of Vuzov. Radiophysics. T. 61. № 10. C. 831– 842. 2018.

– *Kamke E.* Handbook on ordinary differential equations. Moscow: Nauka. 576 c. 1976.

– *Lavrentiev M. A., Shabat B. V.* Methods of the theory of functions of a complex variable. Moscow: Nauka, 736 p. 1973

– *Mareev E. A.* Achievements and prospects of the global electric circuit research // UVN. T. 180. № 5. C. 527– 524. 2010.

– *Mareev E. A., Stasenko V. N., Shatalina M. V., Dementieva S. O., Evtushenko A. A., Svechnikova E. K., Slyunyaev N. N.* Russian research in the field of atmospheric electricity in 2015– 2018 // Izv. RAS. Atmosphere and Ocean Physics. T. 55. № 6. C. 79– 93. 2019.

– *Morozov V. N.* Mathematical modeling of atmospheric-electrical processes taking into account the influence of aerosol particles and radioactive substances. Monograph. St. Petersburg: publishing house of Russian State Hydrometeorological University, 253 p. 2011.

– *Morozov V. N.* Model of unsteady electric field in the lower

Atmosphere // Geomagnetism and Aeronomy. T. 45. № 2. C. 268– 278. 2005.

– *Hays P. B., Roble R. G.* Quasi-static model of global atmospheric electricity. 1. The lower atmosphere // J. Geophys. Res. V. 84. N A7. P. 3291– 3305. 1979.

–*Kalinin A. V., Slyunyaev N. N. N.* Initial-boundary value problems for the equations of the global atmospheric electric circuit // J. Math. Math. Anal. V. 450. Iss.1. P. 112. 2017.

– *Ogawa T.* Fair-Weather Electricity // J. Geophys. Res. V. 90. N D4. P. 5951– 5960.

1985.

Figure captions

Figure 1. Graph of the function$\Phi^{(\text{ст})}/V_{\infty,s}$ as a function of height *h*. Here .$h_m = 100\text{км}, \theta = \theta_s, \varphi = \varphi_s$

Fig. 2. Graphs of the function$\Phi^{(\text{ст})}/V_{\infty,s}$ (dashed line) and the function$\Phi/V_{\infty,s}$ of the formula (P2.1) as a function of the angleθ at a fixed instant of time$t' = 4\pi\sigma_0 t = 2$ at fixed $h, \varphi = \varphi_s$,$\theta_s = \frac{\pi}{3}$. The left graph corresponds to $h = 70\text{км}, \text{правый} h = 100\text{км}.$

Figure 3. Graphs of the functions$\Phi/V_{\infty,s}\text{и}\Phi^{(\text{ст})}/V_{\infty,s}(\text{штриховаялиния})$ at a fixed moment of time$t' = 4\pi\sigma_0 t = 0.05$ (left) and$t' = 4\pi\sigma_0 t = 0.1$ (right) as a function of height *h*.

Figure 4. Plots of the functions$\Phi/V_{\infty,s}$ and$\Phi^{(\text{ст})}/V_{\infty,s}$ (dashed line) at fixed time$t' = 4\pi\sigma_0 t = 0.5$ (left) and$t' = 4\pi\sigma_0 t = 1$ (right) as a function of height *h*.

Figure 5. Integration contour for finding the original function $\bar{\tilde{R}}_n(\sigma, \sigma_{s0}, p)$.

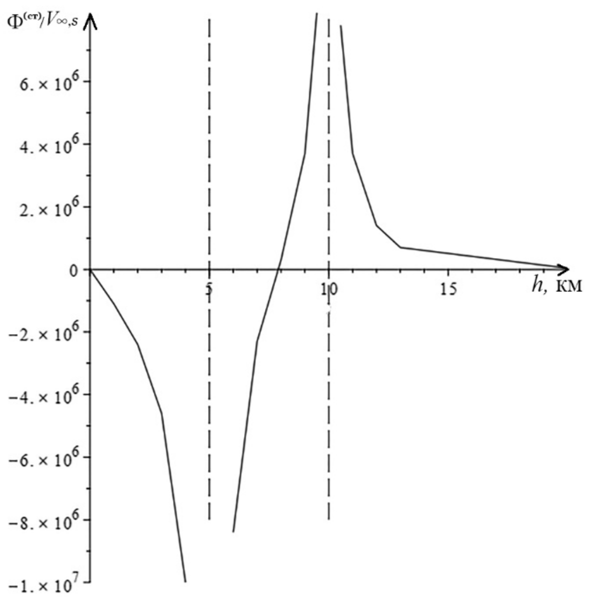

Figure 1.

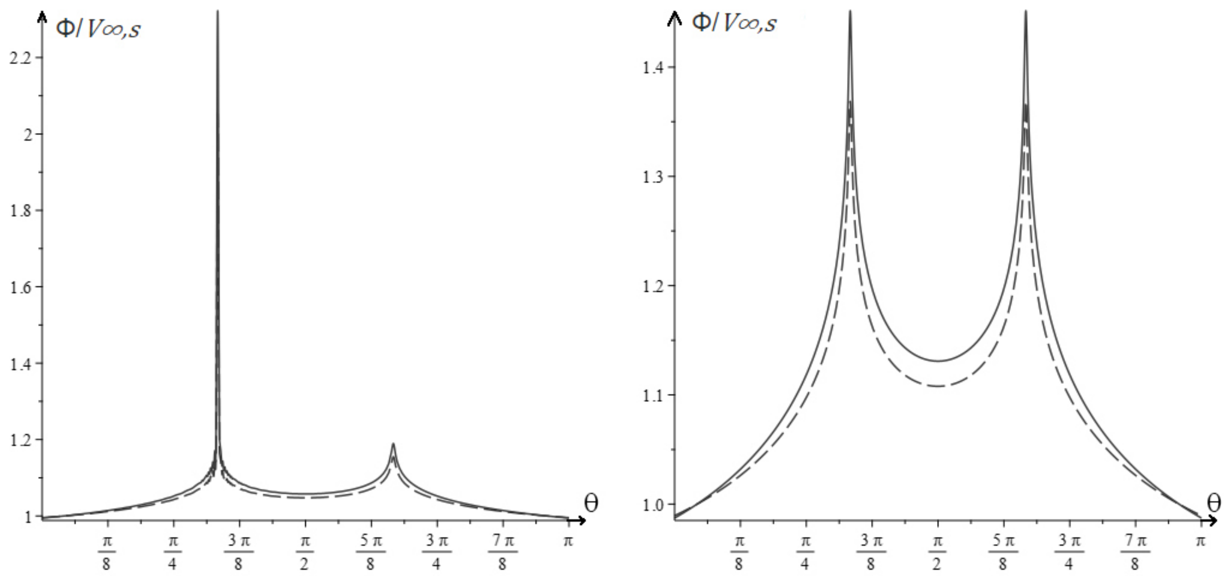

Figure 2.

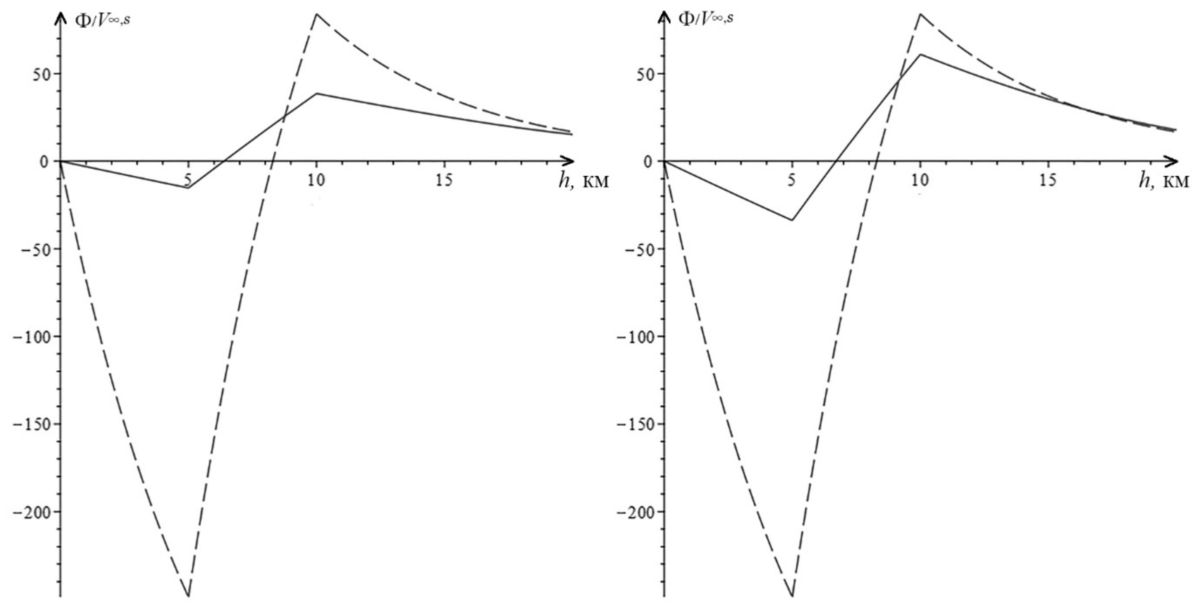

Figure 3.

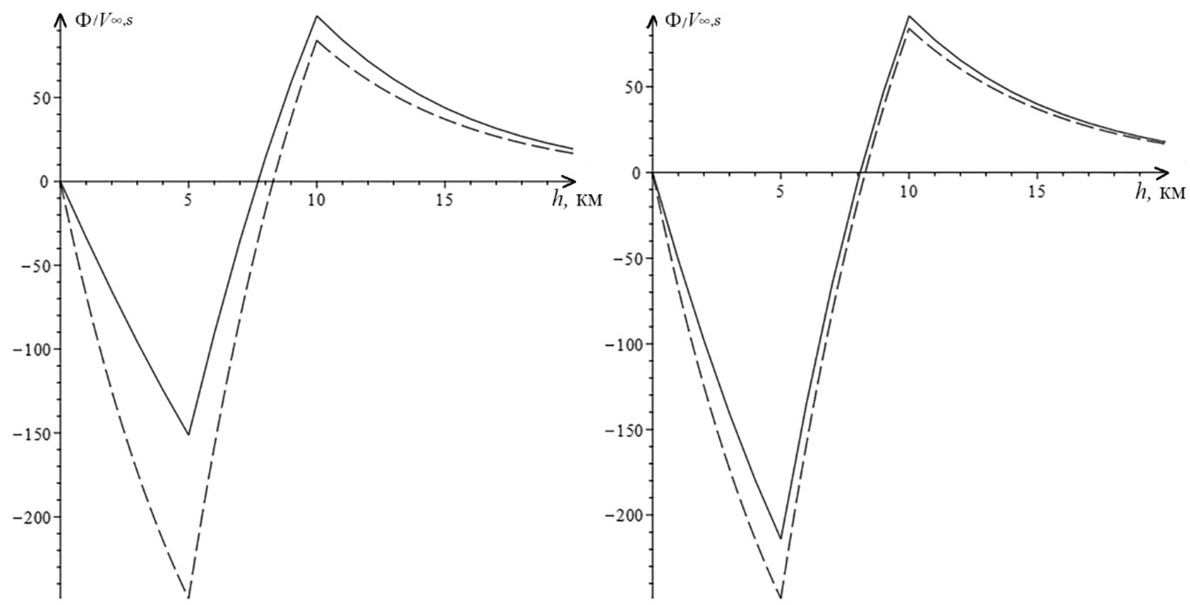

Figure 4.

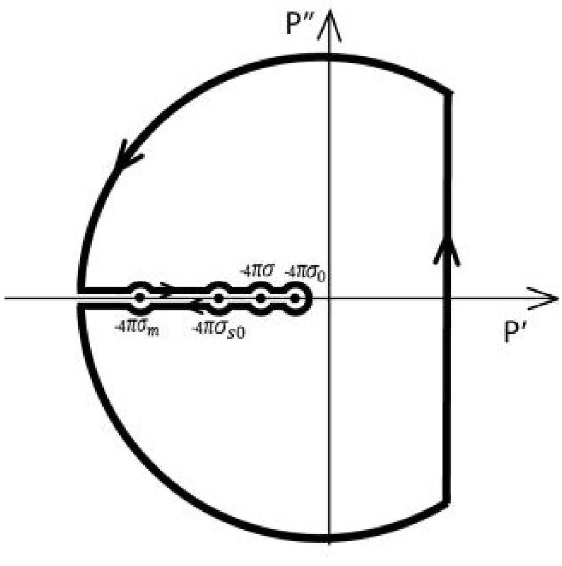

Figure 5.