THE SIMPLE MODEL OF THE EVOLUTION OF MAGNETIC AND KINETIC ENERGY OF GEODYNAMO

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Abstract. The induction and momentum equations are simplified to a dynamical system for the kinetic and magnetic energies in the Earth's core. Stable stationary points of this system give a geomagnetic field of ~10 mT and the cosecant of the angle between the magnetic field vector and the fluid velocity vector is on average about 500 at a known speed of ~1 mm/sec and a generally accepted dynamo power of ~1 TW. With a generally known typical geomagnetic time of the order of a thousand years, harmonic secular variations of the order of several decades and rapid exponential changes of the order of several months, possibly associated with jerks, were obtained. All this is in good agreement with dynamo theory, paleomagnetic reconstructions, numerical modeling and observations. Geomagnetic energy ~10 mJ/kg is four orders of magnitude greater than kinetic energy. Under conditions of such dominance of magnetic energy, an analytical solution was obtained, which over time converges to stable stationary points. Apparently unlikely catastrophes with virtually zero magnetic energy near partially stable stationary points are discussed.

Keywords: *geodynamo, dynamic system, kinetic energy, magnetic energy, magnetic catastrophe*

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1. INTRODUCTION

The full system of geodynamo equations, see for example [Braginsky and Roberts, 1995; Starchenko and Jones, 2002] is extremely complex because it includes the induction equation for the magnetic field vector, the momentum equation for the velocity vector field of a conducting fluid, and essentially the energy equations of heat and mass transfer for entropy and light impurity. From the end of the last century [Glatzmaier and Roberts, 1995] to the present, many numerical models [Christensen et. al, 2010; Bouligand et. al, 2016; Wicht and Sanchez, 2019; Aubert, 2023] have been created that are quite successful in mimicking the known geomagnetic field dynamics on a near-complete system basis. However, the key parameters (and above all the transport coefficients) of all such models differ by many orders of magnitude from their true values, and one has to make extremely distant extrapolations to the real values. Therefore, simplified models are very relevant, among which mean-field models seem to be the most reliable, see, for example, [Krause and Rädler, 1980; Ruzmaikin and Starchenko, 1988]. However, they also remain rather difficult to analyze directly and, at the same time, rely on unproven hypotheses. Simplifications in the spirit of the classical Galerkin method are also used, see, for example, [Vodinchar, 2013; Yushkov and Sokolov, 2018; Moffatt and Dormy, 2019], when the sought quantities (and above all the magnetic field) are approximated on the basis of physical considerations by simple eigenfunctions or similar functions.

The purpose of this work is to create, on the basis of the integral equations of momentum and magnetic induction, the simplest geodynamo-like dynamical system for the total kinetic and magnetic energies. These energies are further expressed through the rms velocity and rms magnetic field, the squares of which are directly proportional to the corresponding specific (in J/kg) energies. Such a representation is capable of reflecting global inversions or excursions of the entire magnetic field if a positively defined RMS magnetic field is assigned a particular sign, so that this quantity remains a continuous and possibly smooth function at zero crossings. In this paper, stationary points are investigated and an analytical geodynamo-like solution is found for the resulting dynamical system for time-fixed parameters.

In the next section, the desired system of equations is derived from the magnetic induction equation and momentum equation. We approximate the energy and other equations of the geodynamo by setting the integral power of the Archimedes buoyancy force work as the commonly accepted value of ~1 TW [Braginsky and Roberts, 1995; Starchenko and Jones, 2002; Aubert, 2023]. A new combined and, in fact, structural parameter L is defined. It is equal to the product of the characteristic size by the characteristic cosecance of the angle between the magnetic field vector and the velocity. By this cosecance we estimate how much the magnetic field is "frozen" in the flow, or, more precisely, how much it is parallel to the flow.

The third section of this paper is devoted to the stable and unstable stationary points of the obtained simplified system for the RMS velocity and magnetic field, which directly reflect the kinetic and magnetic energy. Estimates of physical quantities arising from the analysis of the stationary points are given. All of them are in good agreement with the known modern numerical, theoretical, and, most importantly, observational models of the geomagnetic field. New relations are also obtained.

In the fourth section, the analytical solution of the obtained dynamical system is found under conditions of typical for geodynamo dominance of magnetic energy over kinetic energy. At invariable with time positive power of Archimedes force this analytical solution at any possible initial condition with time asymptotically approaches to the same fixed values, which are given by stable stationary points of the investigated dynamical system. If at sufficiently large turbulent fluctuations the power of the Archimedes force becomes negative for a certain time, it is possible that *the entire* magnetic energy decreases practically to zero. This fundamentally unlikely event can be correlated with previously unexplored global and possibly catastrophic excursions or inversions.

Section 5 summarizes and discusses the main results of this work.

2. DERIVATION OF EQUATIONS

Next, in this section, we derive the simplest dynamical system of two autonomous ordinary differential equations for the kinetic and magnetic energies of the geodynamo. These energies are represented through the rms convection velocity and the rms magnetic field, respectively. The sign of the investigated magnetic field, in this case, can be either positive or negative.

To obtain the equation of evolution of the RMS velocity $u(t)$, let's integrate over the entire volume of the Earth's liquid core the scalar products of the velocity vector $\mathbf{U}$ by all terms of the momentum equation. As a result of identical transformations and neglecting the small terms described below (for details, see, e.g., [Braginsky and Roberts, 1995; Buffett and Bloxham, 2002; Starchenko, 2019]), we obtain

$$\frac{d}{dt}\left(\int_{r_i}^{r_o} \rho \frac{U^2}{2} dV\right) = \int_{r_i}^{r_o}\left(\rho \mathbf{A}\cdot\mathbf{U} - \frac{\mathbf{U}\times\mathbf{B}\cdot\nabla\times\mathbf{B}}{\mu_0} - \rho\nu\left|\nabla\times\mathbf{U}\right|^2\right) dV\,. \tag{1}$$

Here, within the integrals, r_i and r_o denote the radius of the liquid core-solid core and mantle boundary, respectively, ρ is the mean density, and ν is the kinematic viscosity coefficient. The acceleration vector **A is** due to the Archimedes buoyancy force generating convection, which in turn generates a magnetic field with vector **B**. Accordingly, the first term of (1) on the right-hand side is the integral power (generally accepted to be ~1 TW) of the buoyancy force work that generates the geodynamo.

At first glance, it appears that one of the most significant velocity gradients is associated with possibly the narrowest (~1 m thick see, e.g., in [Braginsky and Roberts, 1995]) Ekman layers near the boundaries of the liquid core. In the energy part required here, this effect is accounted for by the last term in equation (1). The corresponding viscous surface integrals [Braginsky and Roberts, 1995; Starchenko, 2019] were neglected in (1) primarily because of gravitational and electromagnetic blocking of the relative rotation of the solid core by the mantle [Dumberry and Mound, 2010]. This blocking causes the difference between the angular velocities of the mantle and the solid core to decrease to such an extent that the energetics of global convective processes essentially dominate the viscous effects in the narrow Ekman layers due to this difference.

Divide (1) by the mass of the liquid nucleus M. We obtain in the left part of (1) directly by definition udu/dt. First on the right is the specific integral power of the Archimedes buoyancy force a, which we consider fixed in time. Next is the specific power of the magnetic Lorentz force, which, relying on the degrees of the constituent velocity and magnetic field, it is natural to estimate as $ub^2/(L\rho\mu_0)$. Here L is the characteristic external scale divided, relying on the corresponding vector product, by the typical sine s of the angle between the velocity and magnetic field vectors.

The compound parameter L, combines the characteristic spatial dimension of the magnetic field l and a measure of the extent to which the magnetic field lines of force are parallel to the current flow lines of the conducting fluid. The value of the cosecance (inverse sine $1/s$) of the angle between the velocity vector and the magnetic field is proposed here as such a measure, which is apparently proportional to some degree of the magnetic Reynolds number. The larger this number is - the more strongly the field is "frozen" into the current, see e.g. [Moffatt, Dormy, 2019]. While the fact that the field is "frozen in" is not equivalent to the parallelism discussed here, surely some connection between the two must exist. The characteristic size of the magnetic field l can be obtained both directly from observations [Starchenko, 2015] and theoretically [Starchenko, 2014, 2019]. Thus, let us finally define the parameter $L = l/s$.

Closing the considered relation (1)$/M$ is the specific integral power of the diffusion force, which it is natural to estimate as $-u^2/T_u$. Thus, in the framework of the developed approach the integrals are represented through their components, to which they are directly proportional. The time T_u is the diffusion time and b is the rms magnetic field. As a result, we obtain the evolution equation for the velocity

$$u\mathrm{d}u/\mathrm{d}t = a - ub^2/(L\rho\mu_0) - u^2/T_u. \qquad (2)$$

Similarly, scalarly multiply both parts of the induction equation by the magnetic field vector, integrate over the volume and get (σ - conductivity):

$$\frac{d}{dt}\left(\int_{r_i}^{r_o}\frac{B^2}{2\mu_0}dV\right)=\int_{r_i}^{r_o}\left(\frac{\mathbf{U}\times\mathbf{B}\cdot\nabla\times\mathbf{B}}{\mu_0}-\frac{|\nabla\times\mathbf{B}|^2}{\mu_0^2\sigma}\right)dV\,. \tag{3}$$

Dividing now all terms of expression (3) by the volume of the liquid core and based on the considerations presented above, we finally obtain the evolution equation for the magnetic field (T_b is the diffusion time for the magnetic field):

$$b\mathrm{d}b/\mathrm{d}t = ub^2/L - b^2/T_b. \tag{4}$$

This equation (4), together with equation (2), constitutes the system sought.

3. STATIONARY POINTS

The stationary points of the above system from (2) and (4) are found in the standard way - we equate the right-hand sides to zero and solve the corresponding algebraic systems. All parameters considered in this section are assumed to be constant. The justification of this assumption is given further in the text after the values of key parameters necessary for such justification are obtained.

We start with the stationary points of the system (2, 4) corresponding to a nonzero magnetic field (index - S):

$$u_S = L/T_b = u_*, \qquad b_S = \pm\{\rho\mu_0[T_b a - L^2/(T_u T_b)]\}^{1/2}. \tag{5, 6}$$

Here $u_*$is a typical velocity. The theoretically and observationally known value of this velocity of 1 mm/s [Braginsky and Roberts, 1995; Starchenko and Jones, 2002; Christensen et. al, 2010; Starchenko, 2019] together with the millennial T_b = 30 Gs [Bouligand, 2016; Starchenko and Yakovleva, 2021; Panovska et. al, 2013] give L = 30 Mm. This is an order of magnitude larger than the radius of the Earth's core, indicating a corresponding and highly significant exceeding of the critical geodynamo level. Following the definition of $L= l/s$ in the previous section and the value of l = 60 km from [Starchenko, 2015] we obtain a typical cosecance of $1/s$ = 500. This very significant value corresponds to the aforementioned significant "parallelism" of the flow and field, as well as to the large magnetic Reynolds number $\mathrm{R_m}= \sigma\mu_0 u_S\, c$ = 3500 (core radius c =3.5 Mm, and magnetic diffusion coefficient $1/\sigma\mu_0 = 1$ m^2/s).

The treatment of the dynamo system through the size L and especially the cosecance $1/s$ corresponds to the previously little discussed mechanism of equilibrium of the RMS value of the magnetic field due to the inverse effect on the flow. Here we can assume that such equilibrium is largely due to the tendency for the magnetic field and flow velocity to become parallel. In this case, no significant suppression of the RMS values of velocity is required to achieve equilibrium, but only a change in the structure of the generated magnetic field is sufficient to make it predominantly parallel to the current velocity everywhere. For the geodynamo this parallelism is huge - $1/s = 500$ (!), which means a very small change in the velocity field compared to the non-magnetic situation and, on the contrary, a huge change in the magnetic field compared to the near-critical level of magnetic field generation. Of course, further justification of this assumption from theoretical and numerical models is required, which will allow us to relate R_m to $1/s$ in particular.

The specific geodynamo power a used here is about 0.3 πW/kg, as shown by Starchenko and Jones [2002], who introduced this parameter. At the same time, a related parameter to a is the total power of the Archimedes force or buoyancy work, which in our notations is aM, which is of the order of 1 TW, and M is the mass of the liquid core. This, generating convection and geomagnetism, total power is known from the very times of origin [Braginsky, 1964; Lowes, 1970; Jacobs, 1975] of geodynamo problems and is used in many (if not all) works concerning energetic aspects of geodynamo.

For the successful generation or rather - for the very existence of a significant stationary magnetic field it is necessary, following from the positivity of the subcorrelated expression in (6) and the above estimates, to fulfill the threshold condition

$$T_u > 3 \text{ Ms}. \tag{7}$$

This condition is most likely fulfilled by a nearly thousand-fold margin, since the turbulent value of the magnetic diffusion time T_b should be comparable to the also turbulent diffusion time T_u [Braginsky and Roberts, 1995; Shebalin, 2018]. Accordingly, the typical rms field $b_* = (\rho\mu_0 T_b a)^{1/2}$ of (6) is quite large, about 10 mTL (100 Gs) in the interior of the Earth's liquid core. This corresponds to the strong-field geodynamo, which was first proposed by Stanislav Iosifovich Braginsky [1964]. In this case, the relative geomagnetic energy $b_*^2/2\mu_0\rho$ is of the order of 10^{-2} J/kg, which is much greater than the relative kinetic energy $u_*^2/2 \sim 10^{-6}$ J/kg. This excess by orders of magnitude is further considered to be a characteristic of a typical geodynamo-like system following [Braginsky and Roberts, 1995; Starchenko and Jones, 2002; Christensen et. al, 2010] and many others. Accordingly, a small parameter equal to the ratio $\mu_0\rho u_*^2/b_*^2$ of the kinetic and magnetic energies is introduced. Obviously, here this parameter is equal to $u_*^2/T_b a$.

Thus, the obtained stationary points from (5-6) are in good agreement with geomagnetic observations, numerical models, and generally accepted provisions of geodynamo theory. In fact, based on the characteristic time of magnetic diffusion T_b, the known migration of force lines with velocity u_*, and a rather confidently estimated power a, we obtain a new structural parameter $L= l/s$, the characteristic magnitude of the magnetic field $b_{(*)}$, the ratio of kinetic and magnetic energies $u_*^2/T_b a$.

However, due to the fact that the real parameters (and first of all a) of the considered system are not stationary, but depend on time - at the first stage it makes sense to investigate the stability of the obtained stationary points. In this case, on some relatively small time interval (actually smaller than T_b) it is quite possible to consider the parameters approximated as constant, so that from a linear system to assess the stability and dynamics of all kinds of small deviations from the stationary points as generally as possible. Let us consider these deviations ε and δ, which are substituted through $u = \varepsilon + u_S$ and $b = \delta + b_S$ in (2, 4). Leaving only small first-order quantities, we obtain the desired linear system

$$u_S \mathrm{d}\varepsilon/\mathrm{d}t = -[(b_S)^2 \varepsilon + 2u_S b_S \delta]/(L\rho\mu_0) - 2u_S \varepsilon/T_u, \quad \mathrm{d}\delta/\mathrm{d}t = (b_S \varepsilon + u_S \delta)/L - \delta/T_b. \qquad (8, 9)$$

This is a linear system of second order, we write its general solution in the following form:

$$\delta = C_+ \exp(k_+ t) + C_- \exp(k_- t), \quad \varepsilon = L(\mathrm{d}\delta/\mathrm{d}t)/b_S, \qquad (10, 11)$$

$$k_\pm = -(T_b^2 a/L^2 + 1/T_u)/2 \pm [(T_b^2 a/L^2 + 1/T_u)^2/4 - 2T_b a/L^2 - 2/(T_u T_b)]^{1/2}. \qquad (12)$$

If the geodynamo is active, then - the specific power of the Archimedean force $a > L^2/(T_u T_b^2)$, see (6), and the real part of (12) is negative. Therefore, we state that the stationary points (5-6) are stable. For small deviations from these points, the system returns to them, reducing the initial deviation by a factor of e in about a few months at the parameter values assumed above, which is consistent with the relatively short geomagnetic periods most possibly manifest in phenomena such as jerks [Aubert and Finlay, 2019]. The imaginary part of (12), on the other hand, gives periodic oscillations with periods of about several decades, which are consistent with the well-known secular geomagnetic variations. All these and the time intervals described above also agree well with direct observations, paleomagnetic reconstructions, numerical modeling, and known theoretical statements, see, for example, [Arneitz et al., 2021; Panovska et al., 2013; Aubert, 2023; Moffatt and Dormy, 2019].

Note that even the longest of the intervals associated with stability is one or two orders of magnitude shorter than the magnetic time T_b, which confirms the initial assumption that it is possible to use time-fixed parameters of the system for relatively short time intervals.

Let us conclude this section with a study of stationary points with zero magnetic field:

$$b_0 = 0,\ u_0 = \pm(T_u a)^{1/2}. \quad (13)$$

The analogous (8-9) system for determining stability takes the simplest form

$$\mathrm{d}\delta/\mathrm{d}t = u_0\delta/L - \delta/T_b\ ,\ \mathrm{d}\varepsilon/\mathrm{d}t = -2\varepsilon/T_u\ . \quad (14,\ 15)$$

Obviously, the variables have separated and equation (15) gives the simplest and obviously stable solution ~exp($-2t/T_u$) for the velocity, and equation (14) gives the solution ~exp($u_0t/L-t/T_b$) for the magnetic field. The latter solution is unstable when the realistic (see earlier in this section) convection velocity $u_0 > L/T_b$ is sufficiently large, and stable when the opposite inequality holds. Such partially stable stationary points may correspond not so much to well-known inversions/excursions (in which the total magnetic energy remains quite significant, see, e.g., [Moffatt and Dormy, 2019; Gwirtz et al., 2021]), but rather to as yet unexplored catastrophes with a near-zeroing of the entire magnetic field.

4. SIMILAR GEODYNAMO SOLUTION

In the above equation (2) for velocity, the orders of magnitude of the term on the left ~u_*^2/T_b and the last term on the right ~u_*^2/T_u almost coincide due to the highly turbulent nature of the currents, in which the magnetic diffusion time T_b is comparable to the diffusion time T_u [Braginsky and Roberts, 1995; Shebalin, 2018]. The second term on the right is of the same order of magnitude as the first term a. Dividing u_*^2/T_b by a, we obtain the ratio of kinetic and magnetic energies, which is extremely small for a geodynamo-like system, see details in the previous section. Therefore, in the main order of the similar geodynamo approximation, the system of equations (2) and (4) is simplified to

$$a = ub^2/(L\rho\mu_0),\ b\mathrm{d}b/\mathrm{d}t = \rho\mu_0 a - b^2/T_b. \quad (16,\ 17)$$

Let us write the general solution of this system in the form (C is the integration constant):

$$u = \rho\mu_0 La/b^2,\ b = \pm[\rho\mu_0 T_b a - C\exp(-2t/T_b)]^{1/2}. \quad (18,\ 19)$$

The initial (at t = 0) value of $b = \pm(\rho\mu_0T_ba - C)^{1/2}$ can be any value, determining the corresponding C given $\rho\mu_0T_ba$ and the sign before the bracket. With time, the magnitude of the rms magnetic field will asymptotically approach the value $\pm[\rho\mu_0T_ba)]^{1/2}$, which corresponds to the stationary points (6) in the considered approximation. Since, in physical essence, the considered magnetic field b is the square root of the total magnetic energy, it is not surprising that at the time-fixed parameters used here, this energy tends to some fixed value. In the framework of such an approach it is impossible to describe the well-known inversions or excursions, since they are primarily oriented to the dipole component, which makes a very small contribution to the total magnetic energy of the geodynamo considered here, see [Glatzmaier and Roberts, 1992]. [Glatzmaier and Roberts, 1995; Braginsky and Roberts, 1995; Starchenko and Smirnov, 2021; Gwirtz et al., 2021].

Reaching the zero field in (19) is formally possible only at t = 0 and $C = \rho\mu_0T_ba$. In this single variant, the field is zero only at the initial moment, and then there is a monotonic growth of b^2 with time. If we assume that for some time interval a became negative - then at $C = \rho\mu_0T_ba$ we get b^2decreasing from some moment $t < 0$ in the past to zero at the moment t = 0. Thus, a kind of "precursor of the global excursion/inversion" can be obtained, when all magnetic energy is practically zero and there is a transition to the "zone of influence" of the partially stable stationary point (13). We speak so far only of the "precursor" because of the singularity for u in (18) present near b = 0.

5. RESULTS AND DISCUSSION

The main result of this work is the construction, on the basis of integral equations, of the simplest dynamical system, apparently, which quite correctly describes such a geodynamo evolution of the total kinetic and magnetic energies. The correctness of this system is shown by using as known input parameters well estimated from observations, numerically and from theory: characteristic velocity u_*~1 mm/s, typical magnetic diffusion time T_b ~ 1 thousand years and total geodynamo power ~1 TW = aM (M - mass of the liquid core).

The product of velocity and time yields a new output parameter L = 30 Mm, which is apparently an optimally combined structural parameter $L= l/s$, combining the characteristic magnetic scale l and the typical sine s of the angle between the velocity and magnetic field vectors. This composite parameter L, combines the characteristic spatial dimension of the magnetic field l and a measure of how far the magnetic field lines of force are parallel to the current flow lines of the conducting fluid. The value of the cosecance (inverse sine $1/s$) of the angle between the velocity vector and the magnetic field is proposed here as such a measure, which is apparently proportional to some degree of the magnetic Reynolds number. The larger this number is, the more strongly the field is "frozen" into the current, see e.g. [Moffatt and Dormy, 2019]. While the fact that the field is "frozen in" is not equivalent to the parallelism discussed here, surely some connection between the two must exist. The characteristic size of the magnetic field l can be obtained both directly from observations [Starchenko, 2015] and theoretically [Starchenko, 2014; 2019]. Thus, let us finally define the parameter $L = l/s$.

The treatment of the dynamo system through the size L and especially the cosecance $1/s$ corresponds to the previously little discussed mechanism of equilibrium of the RMS magnitude of the magnetic field due to the inverse effect on the flow. Here we can assume that such equilibrium is largely due to the tendency for the magnetic field and flow velocity to become parallel. In this case, no significant suppression of the RMS values of velocity is required to achieve equilibrium, but only a change in the structure of the generated magnetic field is sufficient to make it predominantly parallel to the current velocity everywhere. For the geodynamo this parallelism is huge - $1/s = 500$ (!), which means a very small change in the velocity field compared to the non-magnetic situation and, on the contrary, a huge change in the magnetic field compared to the near-critical level of magnetic field generation. Of course, further justification of this assumption from theoretical and numerical models is required, which will allow, in particular, to relate R_m to $1/s$.

The second output parameter is the viscous diffusion time, which, because of the developed turbulence, is simply considered to be of the order of the same 1 thousand years as the magnetic diffusion time.

The following output parameters are related to the stationary points of the system and their stability.

The obtained typical rms field is quite large - about 10 mTl (100 Gs), which corresponds to the strong-field geodynamo, which was first proposed by Stanislav Iosifovich Braginsky [Braginsky, 1964]. In this case, the specific geomagnetic energy ~ 10 mJ/kg is much greater than the specific kinetic energy ~0.001 mJ/kg. Let us introduce a new parameter, say E, for the ratio of kinetic and magnetic energies, which is here equal to $u_*^2/T_b a$. It is suggested that a small $E << 1$ is a characteristic feature of a typical such geodynamo system. Strangely enough, such an assumption does not seem to have been postulated before.

At small deviations from stable stationary points, the system returns to them, reducing the initial deviation by a factor of e in about a few months, which is consistent with the shortest geomagnetic periods here, which are perhaps most pronounced in phenomena such as jerks.

Near the stable stationary points, there are also periodic oscillations with periods of about several decades, which agrees well with the generally known secular geomagnetic variations. All these and the time intervals described above also agree well with direct observations, paleomagnetic reconstructions, numerical modeling, and known theoretical positions, see, for example, [Arneitz et al., 2021; Panovska et al., 2013; Aubert, 2023; Moffatt and Dormy, 2019; Starchenko, 2014].

Thus, the simplest dynamical system obtained from integral equations allows us to physically justify three most important characteristic times at once: diffusion time (about thousand years), time of secular variations (tens of years) and the shortest time (of the order of several months), which can correspond to jerks and possibly to other unexplored phenomena. A new structural parameter is determined, the cosecance $1/s$ of the typical angle between the magnetic field vector and the velocity vector, which characterizes the parallelism of the magnetic field and the flow of the conducting fluid. The product of this cosecance by the characteristic size of the magnetic field gives another new, in fact, critical parameter *L,* which exceeds the radius of the Earth's core by an order of magnitude, indicating that the critical level of excitation of the geodynamo is exceeded quite significantly. Another little-studied but apparently important parameter - the ratio of kinetic and magnetic energies *E*, which is small for systems like the geodynamo - is also actualized. For the geodynamo itself, this parameter is $\sim 10^{-4}$with a characteristic magnetic field of ~10 mTl obtained here.

The analytical solution of the obtained dynamical system is found under conditions of typical for geodynamo dominance of magnetic energy over kinetic energy, when $E \ll 1$. At a stationary in time positive power of Archimedes force *a* this analytical solution at any possible initial condition with time asymptotically tends to constant values, which are given by stable stationary points of the investigated dynamical system.

If the power of the Archimedes force becomes negative for some time as a result of a large and apparently unlikely fluctuation - then a decreasing practically to zero magnetic energy is possible, which can be correlated with global catastrophic excursions/inversions near partially stable stationary points. Physically, the power of the Archimedes force is the first term on the right-hand side of formula (1). This term is determined by the predominantly positive (when the dynamo is running) scalar product of the radial velocity component by the acceleration due to the Archimedes buoyancy force. However, at such highly developed turbulence as in the geodynamo, giant fluctuations leading to negative values of the considered first term (1) are possible. Obviously, such fluctuations are extremely unlikely to realize the reduction of the magnetic energy almost to zero, since they should be very large for this purpose and continuously manifest themselves on a sufficiently long time interval from thousands of years.

It should be emphasized that the obtained system cannot, in principle, directly reflect the well-known excursions or inversions, since they are primarily associated with the dipole component, which is usually several orders of magnitude (in energy) smaller than the total magnetic energy of the geodynamo considered here. At the same time, some energy evolution may be a precursor to a conventional inversion or excursion [Gwirtz et. al, 2021]. However, discussed here, seemingly for the first time, catastrophic, essentially catastrophic inversions/excursions with near-zero magnetic energy may prove incomparably more destructive than all these well-known geomagnetic dipole inversions/excursions.

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